# Control of Flexible Structures with Spillover Using an Augmented Observer

Yossi Chait* and Clark J. Radcliffe†
*Michigan State University, East Lansing, Michigan*

**Modern modal control methods for flexible structures have control and observation spillover that can degrade performance and reduce the stability margin of the closed-loop controlled structure. The sensor output is often filtered to reduce observation spillover, however, the filter introduces signal distortion and perturbs the closed-loop system eigenvalue locations. This perturbation can reduce the stability margin and jeopardize convergence of a deterministic observer. If the filter equations are not explicitly included in the observer design, then the separation principle between the controller and the observer states no longer holds when present in the unfiltered system. A new method is presented where the observer equations are augmented to include a first-order filter dynamics. The separation principle, controllability, and observability of the unfiltered system are invariant to the filter dynamics in this new method, resulting in no perturbation of controlled system eigenvalue locations. The filter cutoff frequency can be located even within the bandwidth of the system, thereby increasing the filter effectiveness in reducing observation spillover. Spillover-generated errors in closed-loop eigenvalues of these control methods are compared using a numerical example.**

## Introduction

VIBRATION control of systems whose dynamics are modeled with partial differential equations has been studied extensively. One current example is large space structures (LSS) whose vibration model consists of an infinite number of lightly damped modes. Finite-dimensional controllers for certain classes of infinite-order systems (IOS) have been derived using the partial differential equation describing the system dynamics.[1,2] These controllers are based on a reduced-order model (ROM), but they take into account the actual IOS. It was noted by Schut[3] that it is difficult to verify the stability of the actual IOS controlled by a ROM-based controller. In LSS practical control design, the only accurate model available is a ROM obtained via a finite-element or an identification procedure, because of the geometrical complexity of the structure.

The interaction between the ROM-based controller and the actual IOS is termed spillover. The effects of spillover on the closed-loop system are to degrade the system performance and, in some cases, cause instability. The spillover negative effects have been demonstrated in theory[1] and in experiments.[4,5] Spillover bounds[1] were utilized in a ROM-based controller design to guarantee in theory the stability of the IOS system. Various approaches have been suggested for reducing spillover effects when accurate information is available beyond the ROM[1,6,7] In some applications where an approximate information is available on the spectrum of the LSS beyond the ROM's, the placement of sophisticated filters can reduce spillover effects.[1]

A new modern modal control design for flexible structures using an augmented observer is developed here. A brief model formulation for a flexible structure dynamics is presented here along with a conventional modal control design (ROM-based) methods. Errors associated with observation spillover and sensor output filtering are then examined. An augmented observer is developed and shown to eliminate filter-induced errors while reducing observation spillover. Finally, spillover-generated errors in closed-loop eigenvalues of these control methods are compared using a numerical example.

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*Graduate Assistant, Department of Mechanical Engineering and Case Center for Computer-Aided Engineering and Manufacturing; currently, Assistant Professor, Department of Mechanical Engineering, University of Massachusetts, Amherst, MA.

†Associate Professor, Department of Mechanical Engineering and Case Center for Computer-Aided Engineering and Manufacturing.

## Brief Model Formulation

The modal equations of motion for a flexible structure can be written as[8]

$$MU_{tt}(t) + DU_t(t) + KU(t) = f(t) \tag{1}$$

where $\{M\}$, $\{D\}$, and $\{K\}$ are the mass, damping, and stiffness matrices, respectively, all of infinite dimension. The vector $[U]$ contains the modes of the system, and the modal forces vector $[f]$ includes control forces and disturbances.

The control forces $F$, generated by $N$ point actuators, are

$$f(t) = BF,\ B = \{b_{ij}\},\ b_{ij} = \phi_i(x_j),$$
$$i = 1, 2, \ldots, \qquad j = 1, 2, \ldots, N \tag{2}$$

where $\Phi = [\phi_i]$ is the eigenfunction set (mode shapes) of the system. The measurements produced with $p$ point sensors are

$$Y(t) = CU,\ C = \{c_{ij}\},\ c_{ij} = \phi_j(x_i),$$
$$i = 1, 2, \ldots, p \qquad j = 1, 2, \ldots, \tag{3}$$

For a ROM-based controller, partition the set of modes $U$ into controlled (modeled) and residual such that $U = [U_c^T, U_r^T]^T$, $U_c = [u_1, \ldots, u_n]^T$, $U_r = [u_{n+1}, \ldots]^T$, and $\Phi = [\Phi_c^T, \Phi_r^T]^T$. The control and observation influence matrices are partitioned into $B = [B_c^T, B_r^T]^T$ and $C = [C_c, C_r]$. For control design purposes, the system is transformed into a state-space form by introducing the state vectors $V_c = [U_c^T, \dot{U}_c^T]^T$ with similar notation for $V_r$; the other matrices are defined as $B_c' = [0^T, B_c^T]^T$, $C_c' = [C_c, 0]$, and $B_r'$ and $C_r'$ follow. Using a normalized set of eigenfunctions[8] and assuming no damping in the system ($D = 0$), the new system matrices are

$$A_c = \begin{bmatrix} 0 & I \\ -\Lambda_c & 0 \end{bmatrix} \tag{4}$$

where $\Lambda_c = \mathrm{diag}[\omega_1^2, \omega_2^2, \ldots, \omega_n^2]$ with similar form for $A_r$, and $\omega_i$ are the natural frequencies of the structure. The transformed equations are[1]

$$\dot{V}_c(t) = A_c V_c(t) + B'_c F(t) \tag{5a}$$

$$\dot{V}_r(t) = A_r V_r(t) + B'_r F(t) \tag{5b}$$

$$Y(t) = C'_c V_c(t) + C'_r V_r(t) \tag{5c}$$

## Conventional Control Methods

### Modern Modal Control

The modern model control method (MMC) uses finite-order (ROM) system matrices ($A'_c$, $B'_c$, $C'_c$) in an effort to control the actual IOS. Suppose that the control law uses a deterministic observer driven by the output error

$$\dot{\tilde{V}}_c(t) = A_c \tilde{V}_c(t) + B'_c F(t) + K[Y(t) - \tilde{Y}(t)],$$
$$A_c \in R^{k \cdot k}, B_c \in R^{k \cdot N}, K \in R^{k \cdot p},$$
$$\tilde{Y}(t) = C'_c \tilde{V}_c(t), C_c \in R^{p \cdot k} \tag{6}$$

where $\tilde{V}_c$ is the estimated controlled modes and $\tilde{Y}$ is the estimated output. A linear state feedback of the observed modes typically used in modern modal control has the form

$$F(t) = G\tilde{V}_c(t), \qquad G \in R^{N \cdot k} \tag{7}$$

Define the observer error as

$$\varepsilon_c(t) = \tilde{V}_c(t) - V_c(t) \tag{8}$$

Differentiate Eq. (8) with respect to time and combine the result with Eqs. (5–7) to obtain the MMC system dynamics in terms of the controlled modes and the observer errors

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{\varepsilon}_c(t) \\ \dot{V}_r(t) \end{bmatrix} = \begin{bmatrix} A_c + B'_c G & B'_c & 0 \\ 0 & A_c - KC'_c & KC'_r \\ B'_r G & B'_r G & A_r \end{bmatrix} \begin{bmatrix} V_c(t) \\ \varepsilon_c(t) \\ V_r(t) \end{bmatrix} \tag{9}$$

A ROM is obtained by neglecting the dynamics of the residual modes

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{\varepsilon}_c(t) \end{bmatrix} = \begin{bmatrix} A_c + B'_c G & B' G_c \\ 0 & A_c - KC'_c \end{bmatrix} \begin{bmatrix} V_c(t) \\ \varepsilon_c(t) \end{bmatrix} \tag{10}$$

The finite-order system (10) can be made controllable and observable by appropriate positioning of the actuators and sensors. The state matrix (10) is in a block triangular form, and so its eigenvalues are those of the lower order matrices $\{A_c + B'_c G\}$ and $\{A_c - KC'_c\}$. This property is called the deterministic *separation principle*.[9] The observer error eigenvalues are placed to the left of the corresponding controlled mode eigenvalues.

The above MMC control design ignores the residual modes. However, the measurements are corrupted by residual modes through the observation spillover term $KC'_r$, and the actuators excite the residual modes through the control spillover term $B'_r G$. If both spillover terms are nonzero, then the eigenvalues of $\{A_c + B'_c G\}$, $\{A_c + KC'_c\}$, and $\{A_r\}$ are perturbed.[1] When the eigenvalues of $\{A_r\}$ lie close to the imaginary axis, excessive spillover can generate instability in the residual modes.[1]

### Sensor Output Filtering

In many applications, the only known model for the dynamics of a complicated structure is a finite-order approximation (ROM) obtained via a finite-element analysis or an identification procedure. Usually, this approximated model contains only the lower modes of the structure dynamics.

When there is enough frequency separation between the controlled (modeled) and the residual modes, sensor output filtering can attenuate the effect of higher (residual) frequencies in the sensor output, thereby reducing observation spillover.[1]

In general, the control design is carried out as in the MMC method, and then an analog or digital filter is implemented. To study the filter's effect on the system (8), we define the filtered sensor outputs $z(t)$ of a linear filter driven by actual system outputs $Y(t)$ as

$$\dot{\omega}(t) = A_\omega \omega(t) + B_\omega Y(t),$$
$$A_\omega \in R^{(s \cdot p) \cdot (s \cdot p)}, B_\omega \in R^{(s \cdot p) \cdot p} \tag{11a}$$

$$z(t) = C_\omega \omega(t), \qquad C_\omega \in R^{p \cdot (s \cdot p)} \tag{11b}$$

These equations allow an arbitrary linear filter of order $s$ for each sensor output. Output filtering forces a change in the observer equation (6), where the filtered sensor output and the estimated output are now the driving terms

$$\dot{\tilde{V}}_c(t) = A_c \tilde{V}_c(t) + B'_c F(t) + K[z(t) - \tilde{Y}(t)] \tag{12}$$

The modern modal control with output filtering (MMCOF) system equations in terms of the controlled modes, observer errors, filter, and residual modes are

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{\varepsilon}_c(t) \\ \dot{\omega}(t) \\ \dot{V}_r(t) \end{bmatrix} = \begin{bmatrix} A_c + B'_c G & B'_c G & 0 & 0 \\ -KC'_c & A_c - KC'_c & KC_\omega & 0 \\ B_\omega C'_c & 0 & A_\omega & B_\omega C'_r \\ B'_r G & B'_r G & 0 & A_r \end{bmatrix} \begin{bmatrix} V_c(t) \\ \varepsilon_c(t) \\ \omega(t) \\ v_r(t) \end{bmatrix} \tag{13}$$

A ROM is obtained by neglecting the dynamics of the residual modes

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{\varepsilon}_c(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} A_c + B'_c G & B'_c G & 0 \\ -KC'_c & A_c - KC'_c & KC_\omega \\ B_\omega C'_c & 0 & A_\omega \end{bmatrix} \begin{bmatrix} V_c(t) \\ \varepsilon_c(t) \\ \omega(t) \end{bmatrix} \tag{14}$$

In order to compare the MMCOF design to the MMC's in Eq. (10), we neglect the filter dynamics yielding

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{\varepsilon}_c(t) \end{bmatrix} = \begin{bmatrix} A_c + B'_c G & B'_c G & V_c(t) \\ -KC'_c & A_c - KC'_c & \varepsilon_c(t) \end{bmatrix} \tag{15}$$

The separation principle between the states $V_c$ and $\varepsilon_c$, present in the original system (10), does not hold in both the filtered system (15) due to the coupling term $-KC'_c$ and the system (14) due to the two coupling terms $-KC'_c$ and $B_\omega C'_c$. These coupling terms result in eigenvalue perturbations of the system (10) from their desired locations. Without the separation principle present, eigenvalue placement of the matrix (14) cannot be achieved with standard methods. For this reason, the filter dynamics should not be ignored during controller design.

In practice, closed-loop system eigenvalues are picked to satisfy performance specifications, and the observer eigenvalues are placed somewhat faster so that the total response is dominated by the response of the slower system eigenvalues. It is suggested that the observer eigenvalues should be two to four times faster than the corresponding eigenvalues of the controlled system.[10] Incorporating higher ratios increases the observer sensitivity to noise and modeling errors. A favorable ratio for a specific system should be determined by experimentation in order to achieve balance between noise, speed of response, and modeling errors. This desired ratio may be changed by filter-exerted eigenvalue perturbations.

The filter-exerted eigenvalue perturbations can be reduced or eliminated by either lead-lag networks,[11] which may be of high order, or by including the filter dynamics in the state equations, and then controllability and observability of the new system must be checked again.

## Modern Modal Control with Augmented Observer

The errors associated with sensor output filtering can be avoided by filtering the difference between the actual and the estimated outputs. This filtered difference is then used to drive the observer. The result is modern modal control with an augmented observer (MMCAO), which includes both a standard observer and a sensor output filter. Redefine the filter as

$$\dot{\omega}(t) = A_\omega \omega(t) + B_\omega[Y(t) - \hat{Y}(t)] \tag{16a}$$

$$z(t) = C_\omega \omega(t) \tag{16b}$$

The modified observer equations contain the new driving term $z(t)$:

$$\dot{\hat{V}}_c(t) = A_c \hat{V}_c + B_c' F(t) + Kz(t) \tag{17}$$

The MMCAO system equations in terms of the controlled modes, observer errors, filters, and residual modes are

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{\varepsilon}_c(t) \\ \dot{\omega}(t) \\ \dot{V}_r(t) \end{bmatrix} = \begin{bmatrix} A_c + B_c'G & B_c'G & 0 & 0 \\ 0 & A_c & KC_\omega & 0 \\ 0 & -B_\omega C_c' & A_\omega & B_\omega C_r' \\ B_r'G & B_r'G & 0 & A_r \end{bmatrix} \begin{bmatrix} V_c(t) \\ \varepsilon_c(t) \\ \omega(t) \\ V_r(t) \end{bmatrix} \tag{18}$$

A ROM is obtained by neglecting the dynamics of the residual modes

$$\begin{bmatrix} \dot{V}_c(t) \\ \dot{\varepsilon}_c(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} A_c + B_c'G & B_c'G & 0 \\ 0 & A_c & KC_\omega \\ 0 & -B_\omega C_c' & A_\omega \end{bmatrix} \begin{bmatrix} V_c(t) \\ \varepsilon_c(t) \\ \omega(t) \end{bmatrix} \tag{19}$$

The state matrix (19) is in a block diagonal form, in contrast to the form of the MMCOF method's state matrix (14). Following a result from Ref. 1, it is shown in Lemma 1 below that control design in the MMCAO method reduces to standard eigenvalue placement of a finite-order matrix (19).

*Lemma 1:* Suppose that either observation spillover ($B_\omega C_r'$) or control spillover ($B_r'G$) is zero, then the eigenvalues of the system (18) are the eigenvalues of the following three matrices:

$$A_\eta \triangleq \begin{bmatrix} A_c & KC_\omega \\ -B_\omega C_c' & A_\omega \end{bmatrix} \qquad \text{(augmented observer)}$$

$$(A_c + B_c'G) \qquad \text{(controlled modes)}$$

$$(A_r) \qquad \text{(residual modes)}$$

*Proof*: Set the observation spillover term in Eq. (18) to zero ($B_\omega C_r' = 0$). Redefine the augmented observer state vector as $\eta \triangleq [\varepsilon_c^T, \omega^T]^T$ with a corresponding state matrix $A_\eta$, and reorder the state vector in Eq. (19) from $[V_c^T, \varepsilon_c^T, \omega^T, V_r^T]^T$ to $[\eta^T, V_c^T, V_r^T]^T$, obtaining

$$\begin{bmatrix} \dot{\eta}t) \\ \dot{V}_c(t) \\ \dot{V}_r(t) \end{bmatrix} = \begin{bmatrix} A_\eta & 0 & 0 \\ * & A_c + B_c'G & 0 \\ * & * & A_r \end{bmatrix} \begin{bmatrix} \eta(t) \\ V_c(t) \\ V_r(t) \end{bmatrix} \tag{20}$$

where * refers to a submatrix defined in the matrix of Eq. (19). The above matrix (20) is in a block triangular form, hence, its eigenvalues are the eigenvalues of the three matrices on the diagonal.

In addition, the MMCAO method preserves the separation principle between the controlled modes $V_c$ and the augmented observer states $\varepsilon_c$ and $\omega$. The design of the composite system (20) can be carried independently for the controlled modes and the augmented observer. The conditions for controllability and observability of the MMCAO system (20) are treated in the following lemma:

*Lemma 2*: If the hypothesis of Lemma 1 is satisfied, the eigenvalues of $A_c$ are distinct, the open-loop system is undamped ($D = 0$), the pair ($A_c$, $B_c'$) is controllable, each sensor is placed away from the system's nodes ($C_c \neq 0$), and a linear first-order filter is used; then the finite-order system consisting of the controlled modes and the augmented observer (20) is controllable and observable.

*Proof*: See Appendix A.

The previous lemma implies that the eigenvalues of the system (20) can be assigned without first-order filter-generated eigenvalue perturbations. *In particular, the filter eigenvalue can be placed even within the ROM's frequency bandwidth.* This advantage at the MMCAO method is demonstrated in the example provided later.

Real systems exhibit damping; in particular, the damping ratio is about 0.005 for large space structures. The result of Lemma 2 can be extended to cases with small damping ratios.

*Theorem*: Suppose that all the hypotheses of Lemma 2 are satisfied, and the system exhibits a small amount of natural damping, then the result of Lemma 2 still holds.

*Proof*: Controllability and observability are preserved under small perturbations.[12]

The filter used in the above theorem and Lemma 2 is a first-order filter; however the drawbacks of the MMCOF method exist regardless of the type of filter used. Demonstration of higher-order filters' effect on the MMCOF eigenvalues are summarized in Appendix B. For simplicity of example, the results for a first-order filter are presented in detail below.

## Example

In this example we compare closed-loop eigenvalues of a vibration controller for a Bernoulli-Euler beam designed with the MMC method,[1] the MMCOF method, and the MMCAO method. This example was taken from Balas[1] and has been used to demonstrate the effect of spillover on the closed-loop stability of a system controlled with the MMC method. For simplicity, a low-pass first-order filter is used here. Consider the equation of motion for the transverse vibration of a simply supported beam[8]:

$$EIw_{xxxx}(x, t) + mw_{tt}(x, t) = p(x, t), x \in 0, L \tag{21}$$

with the boundary conditions

$$w(0, t) = w(L, t) = w_{xx}(0, t) = w_{xx}(L, t) = 0 \tag{22}$$

and to simplify the calculations set $E$, $I$, $m$, and $L$ to unity. The mode shapes (eigenfunctions) were $\phi_k(x) = \sin(k\pi x)$, and the natural frequencies were $\omega_k = (k\pi)^2$. There was no natural damping or external disturbance included in the beam model.

The beam was controlled with a single point actuator at $x = 1/6$, and the displacement was measured with a single point sensor at $x = 5/6$. The system was assumed deterministic, and no actuator or sensor dynamics were included. The feedback controller was designed for the first three modes of the beam ($k = 1, 2, 3$). The state observer was a full-order Luenberger observer. As in Ref. 1, the residual system included only the fourth mode ($k = 4$) in this simple example. The control design objective for all methods was to assign the same closed-loop eigenvalues as given by Balas[1] (Table 1).

**Table 1 Open-loop and desired closed-loop eigenvalue locations**

| | Mode | Open loop | Closed loop |
|---|---|---|---|
| Controlled modes | 1 | $0.0 \pm j9.87$ | $-0.7885 \pm 9.87$ |
| | 2 | $0.0 \pm j39.48$ | $-1.3676 \pm j39.48$ |
| | 3 | $0.0 \pm j88.83$ | $-1.5802 \pm j88.83$ |
| Observer errors | 1 | —— | $-1.9710 \pm j7.65$ |
| | 2 | —— | $-3.4190 \pm j30.60$ |
| | 3 | —— | $-3.9510 \pm j68.84$ |

The gain matrices were calculated using the Moore[13] method.

The eigenvalues of the three control methods in the absence of spillover are shown in Table 2. Spillover was eliminated by setting the observation spillover matrices $KC_r'$ in Eq. (9) or $B_\omega C_r'$ in Eqs. (13) and (18) to zero. As expected, the eigenvalue of the residual mode is unaffected by all control methods, and the system eigenvalues are not affected by the residual mode.

The filter produced delay and magnitude attenuation in the frequency response of the MMCOF system. As a controlled mode natural frequency approached the filter cutoff frequency, the distortion produced by the filter yielded larger eigenvalue perturbation. For example, in the MMCOF method with a filter cutoff frequency of 100 rad/s, the first mode stability margin was reduced from a desired value of 0.79 to 0.74, and the third mode stability margin was reduced from 1.58 to 0.86 (Table 2).

A low-pass filter cutoff frequency is normally placed above the highest controlled mode natural frequency to minimize the filter effect on the controlled modes, and as far below the residual mode natural frequency as possible to maximize attenuation of the residual mode in the measurement. In our example, the filter cutoff frequency was restricted to between a third mode natural frequency of 90 rad/s and a fourth mode natural frequency of 158 rad/s.

The MMCAO with a first-order filter always produced identical closed-loop eigenvalues to those of the MMC without filter. This insensitivity of the MMCAO method to output filtering allowed almost arbitrary placement of the filter cutoff frequency, even as low as 20 rad/s, much below the second and third mode natural frequencies (Table 2). Even in this extreme case, no change was observed in the MMCAO controlled system stability margin. In contrast, the MMCOF's first mode stability margin was reduced from 0.79 to only 0.10 when a filter cutoff frequency this low was used.

The change in time constant ratios between controlled modes and their corresponding observer errors was an additional potential problem of the MMCOF method. The time constants were designed to be 2.5 times following recommended practice.[10] Neither first-order filter used in the MMCAO method changed this design ratio. In contrast, when either was used in the MMCOF method, the third mode's ratio was changed (Table 3).

The effects of spillover on the closed-loop system were studied by including the observation spillover matrices in the analysis. Spillover perturbed the closed-loop eigenvalues[1] and, in this example, destabilized the fourth mode eigenvalue when either the MMCAO or MMCOF method was used (Tables 4 and 5). As expected, incorporating a filter reduced observation spillover.

A surprising result was that the MMCOF method generated smaller residual mode perturbation than the MMCAO's generated perturbation. Reasons for the smaller perturbations were: different eigenvalues of these methods before spillover was applied (Table 2), different internal structure of the matrices, and different gain matrix required for the augmented observer $F$ (Appendix A) compared with that for the standard observer $K$. The MMCAO method compensated for the filter distortion with a change in the augmented observer gains so that desired eigenvalue locations would be maintained. As the filter cutoff frequency was made smaller, the magnitude of the MMCAO observer gain matrix became larger. Therefore, the observation spillover coefficient[1] of the MMCAO method was larger than the MMCOF's coefficient, yielding larger eigenvalue perturbations. In this example, the MMCAO observation spillover coefficient was 1.9 times larger than that of the MMCOF method. The reduced MMCOF spillover coefficient was at the expense of smaller effective feedback gains and reduced stability margin of the controlled modes.

**Table 2 Closed-loop eigenvalues—no spillover case**

| | Mode | MMC and MMCAO | MMCOF, $A_\omega = -100$ | MMCOF, $A_\omega = -20$ |
|---|---|---|---|---|
| Controlled modes | 1 | $-0.7885 \pm j9.87$ | $-0.7423 \pm j9.92$ | $-0.5686 \pm 10.03$ |
| | 2 | $-1.3676 \pm j39.48$ | $-1.1652 \pm j40.01$ | $-0.2121 \pm j40.01$ |
| | 3 | $-1.5802 \pm j88.83$ | $-0.8604 \pm j89.54$ | $-0.1026 \pm j89.15$ |
| Observer errors | 1 | $-1.9710 \pm j7.65$ | $-2.1298 \pm j7.58$ | $-2.8263 \pm j7.46$ |
| | 2 | $-3.4190 \pm j30.60$ | $-3.1602 \pm j30.42$ | $-3.1671 \pm j29.75$ |
| | 3 | $-3.9510 \pm j68.84$ | $-4.9843 \pm j67.77$ | $-6.4214 \pm j68.60$ |
| Residual | 4 | $0.0 \pm j157.91$ | $0.0 \pm j157.91$ | $0.0 \pm j157.91$ |

**Table 3 Time constant ratio changes due to output filtering —no spillover case**

| Mode | Desired ratios | MMCAO and MMC | MMCOF $A_\omega = -100$ | MMCOF $A_\omega = -20$ |
|---|---|---|---|---|
| 1 | 2.5 | 2.5 | 2.9 | 5.0 |
| 2 | 2.5 | 2.5 | 2.7 | 14.9 |
| 3 | 2.5 | 2.5 | 5.8 | 62.6 |

**Table 4 Closed-loop eigenvalues—spillover present and $A_\omega = -100$**

| | Mode | MMC | MMCAO | MMCOF |
|---|---|---|---|---|
| Controlled modes | 1 | $-0.7895 \pm j9.97$ | $-0.7885 \pm j9.87$ | $-0.7431 \pm j9.92$ |
| | 2 | $-1.3679 \pm j39.49$ | $-1.3680 \pm j39.48$ | $-1.1605 \pm j40.02$ |
| | 3 | $-1.5810 \pm j88.81$ | $-1.5812 \pm j88.81$ | $-0.8700 \pm j89.54$ |
| Observer errors | 1 | $-1.9656 \pm j7.66$ | $-1.9927 \pm j7.64$ | $-2.1229 \pm j7.58$ |
| | 2 | $-3.4030 \pm j30.62$ | $-3.4135 \pm j30.57$ | $-3.1408 \pm j30.43$ |
| | 3 | $-4.1466 \pm j68.89$ | $-4.1769 \pm j68.84$ | $-5.0975 \pm j67.90$ |
| Residual | 4 | $+0.1766 \pm j157.90$ | $+0.1068 \pm j157.84$ | $+0.0454 \pm j157.83$ |

Direct comparison of MMCOF and MMCAO closed-loop eigenvalue perturbations can be accomplished only with designs that have comparable controlled mode eigenvalues. Since the MMCOF method does not allow standard eigenvalue placement, the MMCAO's designed closed-loop eigenvalues (Table 1) were changed to the MMCOF's eigenvalues (Table 4), while maintaining desired observer time constant ratios of 2.5. The resulting (MMCAO′ Table 5) control and observer gains were smaller and yielded similar residual eigenvalue perturbation in both the MMCOF and MMCAO methods, whereas the MMCOF did not meet closed-loop observer design specifications.

## Conclusions

Modern modal control methods based on a ROM of the dynamics of a flexible structure result in spillover. To reduce spillover effects, it is common to filter the sensor output (MMCOF method). The filter attenuates the effect of the truncated modes in the sensor output; however, it also distorts the sensor output yielding perturbation of closed-loop system eigenvalues from their desired locations. The perturbations generated by the filter can result in reduced stability margin of the controlled modes, and can change the desired time constant ratio between the controlled and observer eigenvalues. This ratio change can jeopardize observer convergence and its robustness in the presence of noise and modeling errors. Because of these perturbations, the control design may not meet the design specifications.

To overcome this difficulty, an augmented observer system was developed that explicitly includes the filter in the observer. In the augmented observer design (MMCAO method), the observer is driven by a filtered output error instead of the difference between the estimated output and the filtered output. In the absence of spillover, it was shown that closed-loop eigenvalues of the MMCAO system can be assigned using standard eigenvalue placement methods and are not affected by the filter dynamics, in contrast to the MMCOF system. The filter cutoff frequency can be located even below the bandwidth of the ROM, thereby increasing the filter effectiveness in reducing observation spillover. Real-time implementation of the MMCAO method is similar to the MMCOF's; however, the MMCAO's observer is augmented with additional states required to implement the filter.

The MMCAO method explicitly treats antispillover lowpass filtering and offers significant advantages over conventional modal methods for control of flexible structures. In the example, the MMCAO reduced spillover generated perturbation of the residual mode eigenvalue location and produced negligible changes in the controlled mode and observer error eigenvalue locations. The MMCAO was shown to have the ability to design for specification in the presence of spillover. Spillover remains a central issue in flexible structures control; however, the MMCAO method presented here minimizes spillover's impact on closed-loop eigenvalues.

## Appendix A

*Lemma A1*: If the hypotheses of the theorem are satisfied and displacement sensors are used, then all of the entries of the matrices $B_c$ and $C_c$ are nonzero.

*Proof*: See Balas.[1]

*Proof of Lemma 2*: By definition, if the system (20) is controllable and observable, then its eigenvalues can be arbitrarily assigned with the gain matrices. Because the matrix (20) has a block triangular form and by hypothesis the pair $(A_c, B_c')$ is controllable, then the proof is now reduced to showing that the eigenvalues of the augmented observer $A_\eta$ can be arbitrarily assigned. Redefine the augmented observer matrix $A_\eta$ as

$$A_\eta = \begin{bmatrix} A_C & K \\ -B_\omega C_c' & A_\omega \end{bmatrix} = \begin{bmatrix} A_c & 0 \\ -B_\omega C_c' & 0 \end{bmatrix} + \begin{bmatrix} KC_\omega \\ A_\omega \end{bmatrix} [0\ I] \triangleq A_1 + FC_1 \tag{A1}$$

where $A_1 \in R^{(k+p)\cdot(k+p)}$, $F \in R^{(k+p)\cdot p}$, $(c_1 \in R^{p\cdot(k+p)}$ $(s=1)$. The eigenvalues of the matrix $A_\eta$ can be assigned arbitrarily if the pair $(A_1, C_1)$ is observable. Before we proceed to form the observability matrix, let us observe the special structure of $A_1$. For brevity, define $E = -B_z C_c'$, $E \in R^{p\cdot k}$. The various powers of $A_1$ have the form

$$A_1^i = \begin{bmatrix} A_C^i & 0 \\ E(A_c)^{i-1} & 0 \end{bmatrix}, \qquad i = 1, 2 \ldots, \tag{A2}$$

and the various powers of $A_c$ have the form

$$A_c^i = \begin{cases} \begin{bmatrix} 0 & (-\Lambda_c)^k \\ (-\Lambda_c)^j & 0 \end{bmatrix} & i \text{ odd}, i \neq 0 \quad k = (i-1)/2, j = (i+1)/2 \\ \begin{bmatrix} (-\Lambda_c)^j & 0 \\ 0 & (-\Lambda_c)^j \end{bmatrix} & i \text{ even}, i \neq 0, j = i/2 \end{cases} \tag{A3}$$

When displacement sensors are used, the matrix $E$ has the form

$$E = -B_z C_c' = -B_z\{C_c 0\} = \{e0\} \tag{A4}$$

where the submatrix $e \in R^{p\cdot(k/2)}$ contains only nonzero elements. We are now ready to construct the observability matrix that has the general form

$$Q = \begin{bmatrix} C_1 \\ C_1 A_1 \\ C_1 A_1^2 \\ \vdots \\ C_1 A_i^j \end{bmatrix}, \qquad j = k + p - 1 \tag{A5}$$

Table 5 Closed-loop eigenvalues—spillover present and $A_\omega = -20$

| | Mode | MMCAO | MMCOF | MMCAO′ |
|---|---|---|---|---|
| Controlled modes | 1 | $-0.7886 \pm j9.86$ | $-0.5691 \pm j10.03$ | $-0.5687 \pm j9.87$ |
| | 2 | $-1.3702 \pm j39.48$ | $-0.2113 \pm j40.01$ | $-0.2121 \pm j39.48$ |
| | 3 | $-1.5815 \pm j88.81$ | $-0.1026 \pm j89.14$ | $-0.1026 \pm j88.83$ |
| Observer errors | 1 | $-2.1333 \pm j7.62$ | $-2.8128 \pm j7.47$ | $-1.4322 \pm j7.65$ |
| | 2 | $-3.5705 \pm j30.58$ | $-3.1529 \pm j29.75$ | $-0.5332 \pm j29.74$ |
| | 3 | $-4.2396 \pm j68.86$ | $-6.4227 \pm j68.66$ | $-0.2756 \pm j68.84$ |
| Residual | 4 | $+0.0695 \pm j157.87$ | $+0.0013 \pm j157.89$ | $-0.0041 \pm j157.91$ |

Combining Eqs. (A2–A4) yields

$$Q = \begin{bmatrix} 0 & I \\ E & 0 \\ EA_c & 0 \\ EA_c^2 & 0 \\ EA_c^3 & 0 \\ EA_c^4 & 0 \\ EA_c^5 & 0 \\ . & 0 \\ . & 0 \\ . & 0 \\ EA_c^{j-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I \\ e & 0 & 0 \\ 0 & e & 0 \\ -e\Lambda_c & 0 & 0 \\ 0 & -e\Lambda_c & 0 \\ e\Lambda_c^2 & 0 & 0 \\ 0 & e\Lambda^2 & 0 \\ . & . & 0 \\ . & . & 0 \\ . & . & 0 \\ . & . & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ T & 0 \end{bmatrix} \tag{A6}$$

where $I \in R^{p \cdot p}$ (identity matrix), $T \in R^{(k \cdot p) \cdot k}$, and $j = k + p - 1$. From this point the proof becomes very complicated for $p > 1$, and we proceed with the proof for $p = 1$ (one measurement), although the following derivation holds for $p \geqslant 1$. The case $p > 1$ will be discussed at the end of this proof. Using the diagonal form of $\{\Lambda_c\}$ and $p = 1$, the matrix $\{T\}$ can be decomposed into $T = T_1 T_2 T_3$, where

$$T_1 = \begin{bmatrix} 1 & \dots & 1 & & 0 & \\ & 0 & & 1 & \dots & 1 \\ \omega_m & \dots & \omega_1 & & 0 & \\ & 0 & & \omega_m & \dots & \omega_1 \\ & . & & . & & \\ & . & & . & & \\ & . & & . & & \\ & . & & . & & \\ & . & & . & & \\ \omega_m^n & \dots & \omega_1^n & . & & \\ & 0 & & \omega_m^n & \dots & \omega_1^m \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & & e_m & & 0 & \\ & . & & & 0 & \\ & . & & & . & \\ & . & & & . & \\ & . & & & . & \\ e_1 & & 0 & & 0 & \\ & 0 & & 0 & & e_m \\ & 0 & & & & \\ & & & & & . \\ & . & & & . & \\ & . & & . & & \\ & 0 & & e_1 & & 0 \end{bmatrix} \tag{A7}$$

$T_3 = \text{diag}\,[1, 1, -1, -1, 1, 1, \dots]$, the signs on the diagonal elements of $\{T_3\}$ are assigned from Eq. (A3), the integer $n$ is defined from Eq. (A3), $[e] = [e_1, \dots, e_m]$, $m = (k/2)$, $j = k + p - 1$, and $T_1$, $T_2$, and $T_3 \in R^{k \cdot k}$.

Since all the elements of $[e]$ are nonzero, it is clear that Rank $\{T_2\} = k$; the special form of $\{T_3\}$ implies that Rank $\{T_3\} = k$. Using the hypothesis that $\omega_i \neq \omega_j$ for all $i$ and $j$ (i.e., distinct eigenvalues), then Rank $\{T_1\} = k$ by the Vandermonde Determinant Theorem.[14] Using simple matrix algebra Rank $\{T\} \triangleq$ min [Rank $\{T_1\}$, Rank $\{T_2\}$, Rank $\{T_2\}$] $= k$. Hence, Rank $\{Q\}$ = Rank $\{T\}$ + Rank $\{I\} = k + p$. By definition, observability of $(A_1, C_1)$ is equivalent to Rank $\{Q\} = k + p$ □.

In the proof for $p > 1$, the requirement is the same: Rank $\{Q\} = k + p$. Showing full rank is based on the structure of the new $T$ matrix that consists of $(k \cdot p)$ rows with at least $k$ independent rows equal to the $T$ matrix in Eq. (A7).

## Appendix B

The purpose of this appendix is to demonstrate that most types of commonly used filters generate eigenvalue perturbations in the MMCOF system. We use here three types of filters that represent most classes of linear filters. The first type is a standard first-order with a cutoff frequency of 100 rad/s:

$$G_1(s) = \frac{100}{s + 100} \tag{B1}$$

A more sophisticated filter is represented by a third-order Chebyshev with 0.5 dB ripple and a cutoff frequency of 130 rad/s (Ref 15):

$$G_2(s) = \frac{1572.37928}{0.001s^3 + 0.16288s^2 + 25.93973s + 1572.37928} \tag{B2}$$

Although a phase-lock-loop (PLL) filter was suggested in Ref. 1 as the most promising method of eliminating observation spillover, it will be shown below that this type of filter used in the MMCOF method produces similar undesired eigenvalue perturbations. PLL filters often suffer from slow transient response, and care must be exercised in their design.[16] Even though PLL filters are typically used when the system natural frequencies are not known exactly, we can model their steady-state dynamics by a group of narrow band-pass (comb) filters. These filters are three band-pass filters in cascade; each is characterized by a second-order Butterworth response with a center frequency equal to that of the controlled mode natural frequency, and a bandwidth of 1% of the center frequency[15]

$$G_3(s) = \frac{0.1s}{s^2 + 0.1s + 97.42} \cdot \frac{0.4s}{s^2 + 0.4s + 1558.51} \cdot \frac{0.9s}{s^2 + 0.9s + 7890.06} \tag{B3}$$

The effects of the filters described by Eqs. (B1–B3) on the MMCOF closed-loop system (13) are shown in Table B1. Table B1 results indicate that as the order of the filter is increased (i.e., a more sophisticated filter), the filter capability to reduce observation spillover is increased, resulting in smaller residual mode eigenvalue perturbation. However, as shown earlier, the desired eigenvalues for the controlled modes and the observer errors as well as the desired time constant ratios cannot be achieved with the MMCOF method regardless of the particular filter used. As the order of the

**Table B1 MMCOF closed-loop eigenvalues—spillover present**

| | Mode | First order | Third order | Band pass |
|---|---|---|---|---|
| Controlled modes | 1 | $-0.7431 \pm j9.92$ | $-0.6980 \pm j9.97$ | $-0.0500 \pm j9.87$ |
| | 2 | $-1.1605 \pm j40.02$ | $-0.7920 \pm j40.50$ | $-0.2000 \pm j39.48$ |
| | 3 | $-0.8700 \pm j89.54$ | $+0.5057 \pm j89.97$ | $-0.4500 \pm j88.83$ |
| Observer errors | 1 | $-2.1229 \pm j7.58$ | $-2.3055 \pm j7.49$ | $-2.9224 \pm j9.29$ |
| | 2 | $-3.1408 \pm j30.43$ | $-2.9173 \pm j30.20$ | $-3.7928 \pm j29.60$ |
| | 3 | $-5.0975 \pm j67.90$ | $-6.3766 \pm j66.85$ | $-6.4580 \pm j69.29$ |
| Residual | 4 | $+0.0454 \pm j157.83$ | $+0.0227 \pm j157.93$ | $-0.0000 \pm j157.91$ |

filter is increased, stability margins become smaller, and time constant ratio changes are increased.

The above choice of the filter's order and specifications was somewhat arbitrary; however the MMCOF drawbacks (Table B1) are regardless of the type of filter used.

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